Linear spaces of tilings

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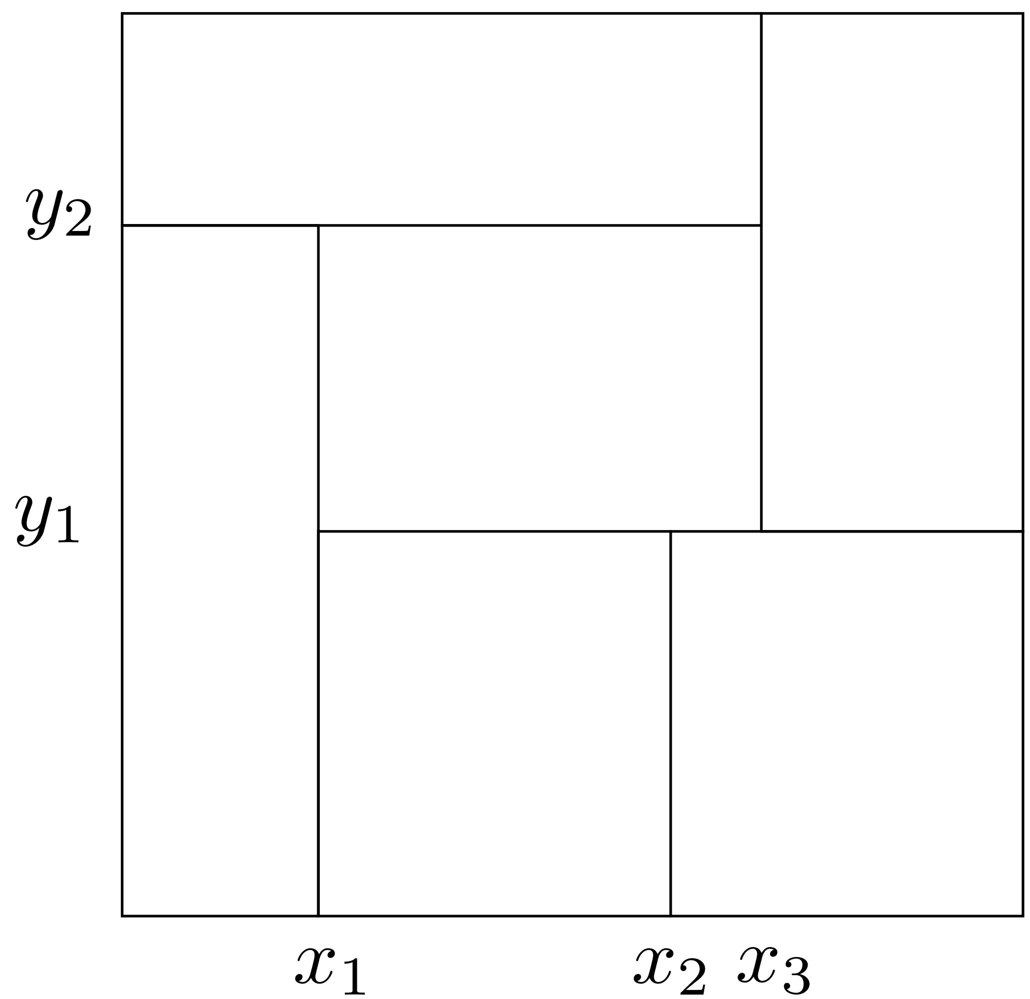
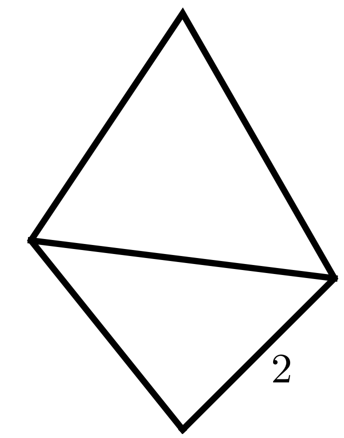
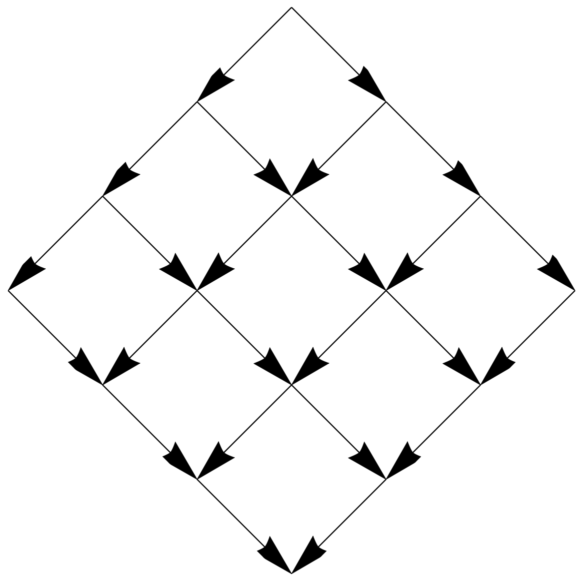
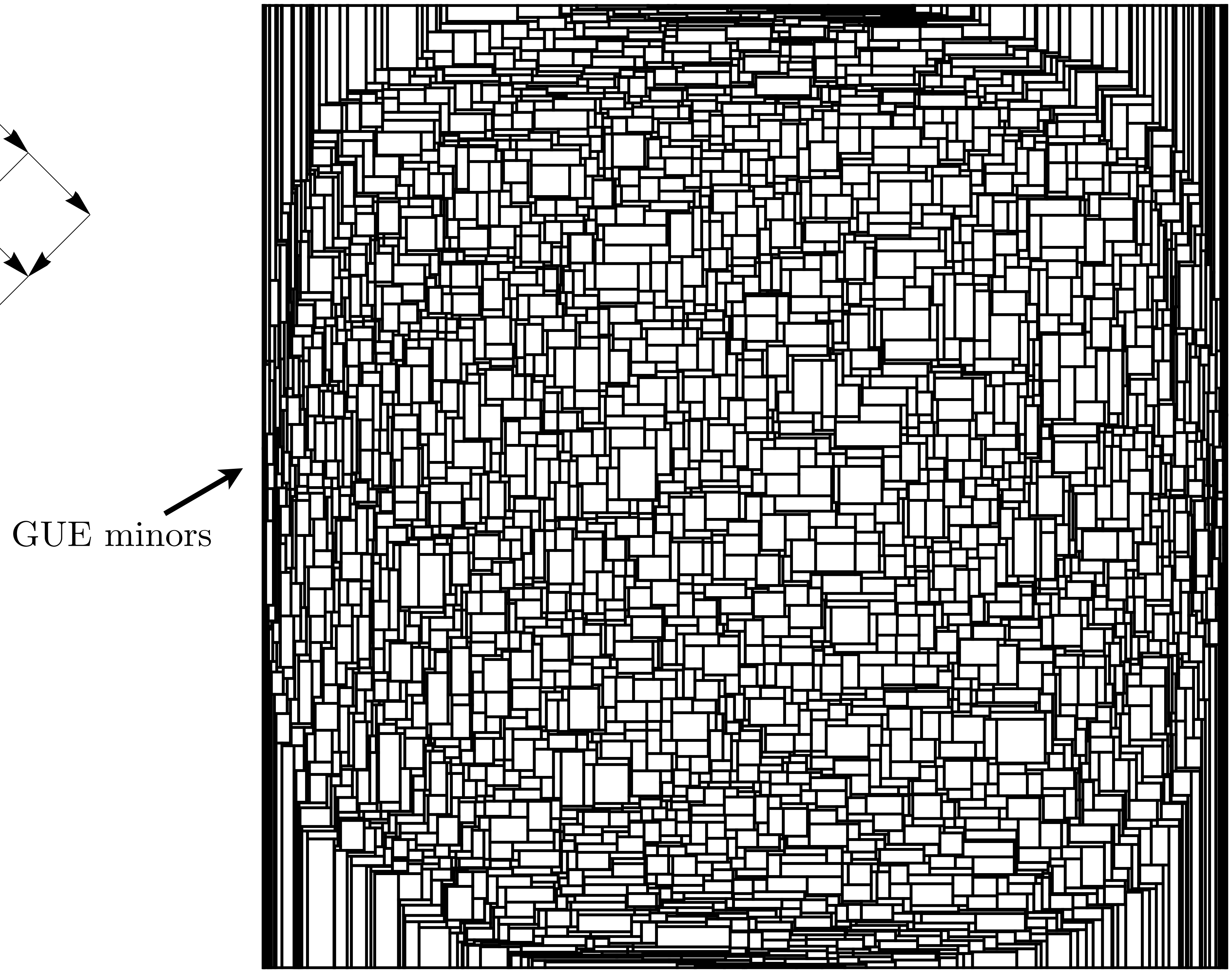
Rectangle tilings come in linear families (polytopes)





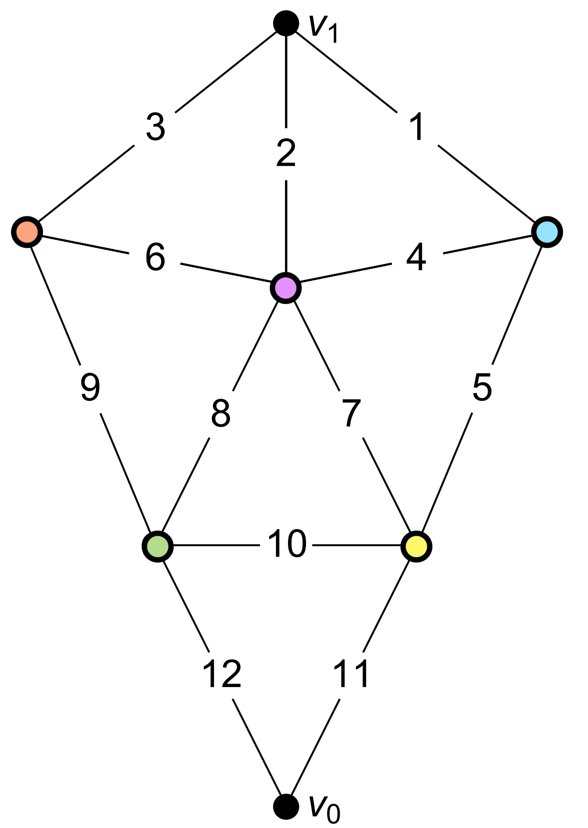
$$P \quad \begin{cases} y_1 < y_2 \\ x_1 < x_2, x_3 \end{cases}$$

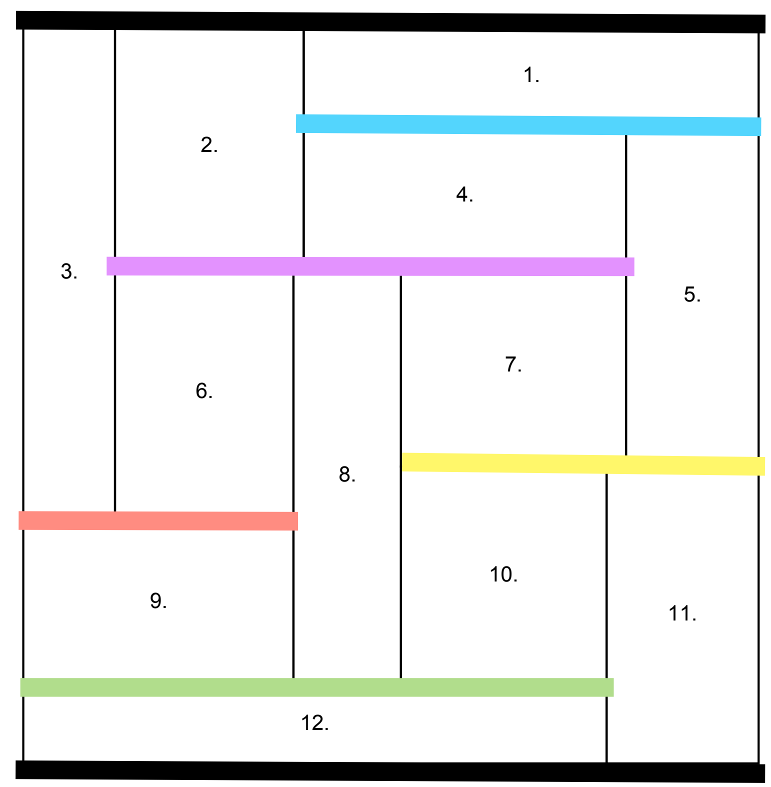




Given such a polytope, one can make a random tiling by choosing a Lebesgue random point

Smith diagram of a planar network [BSST 1939]
(with a harmonic function)

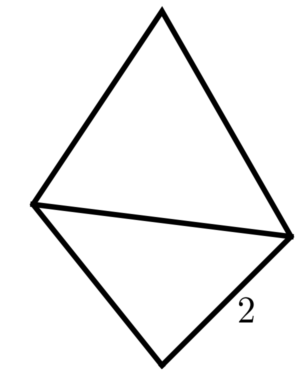




vertex = horizontal line
voltage = y-coordinate
edge = rectangle
current = width
conductance = aspect ratio (width/height)
energy = area

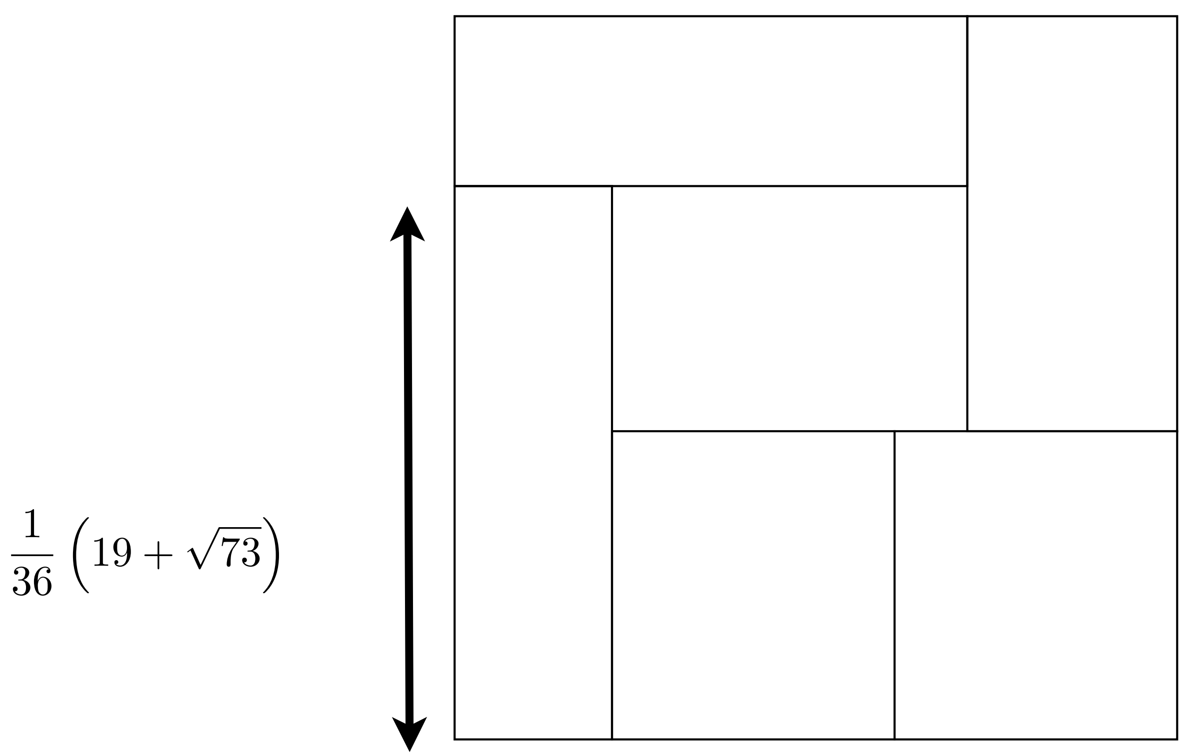
As we change conductances, the polytope can change:
the polytope is defined by *direction of current flow* in the network

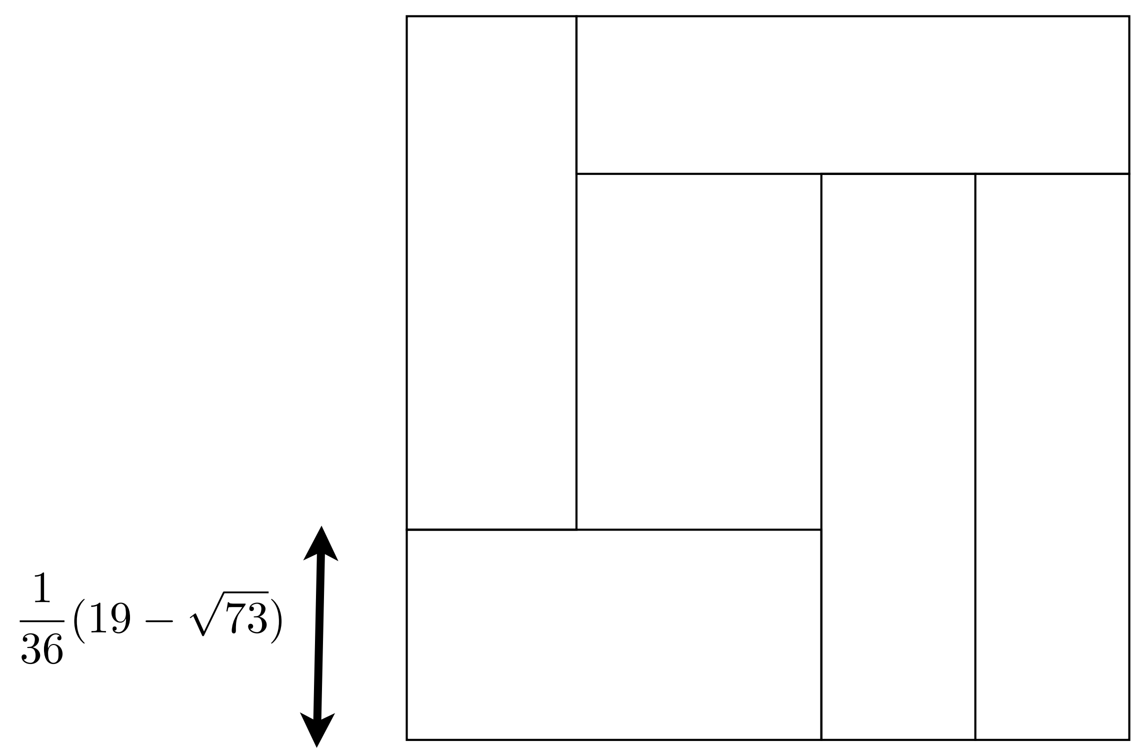
These directions form a *bipolar orientation* of the network.



[K,Abrams]

Thm: There is one fixed-area rectangulation for each bipolar orientation.





$$
\begin{tikzcd}
\underset{\text{conductances}}{(0,\infty)^E} \arrow[rr, "|J_{log}| = 1", "m \text{ to } 1"'] \arrow[dr] & & \underset{\text{energies/areas}}{[0,\infty)^E} \\
& \underset{\text{union of polytopes}}{\{x, y\}} \arrow[ur, "\text{“homeo”}"', "J = K"'] &
\end{tikzcd}
$$

[K,Abrams]

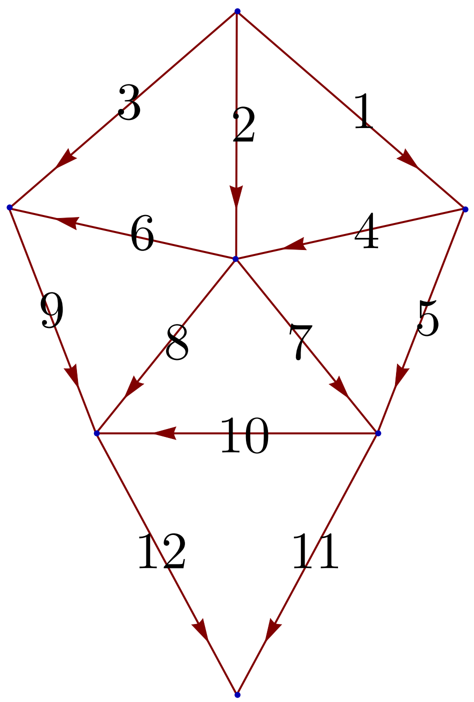
Thm: There is one fixed-area rectangulation for each bipolar orientation.

The corresp. harmonic functions are the solutions of the *enharmonic eqn*:

$$\sum_{u \sim v} \frac{1}{f(v) - f(u)} = 0$$

$$f(v_0) = 0$$

$$f(v_1) = 1.$$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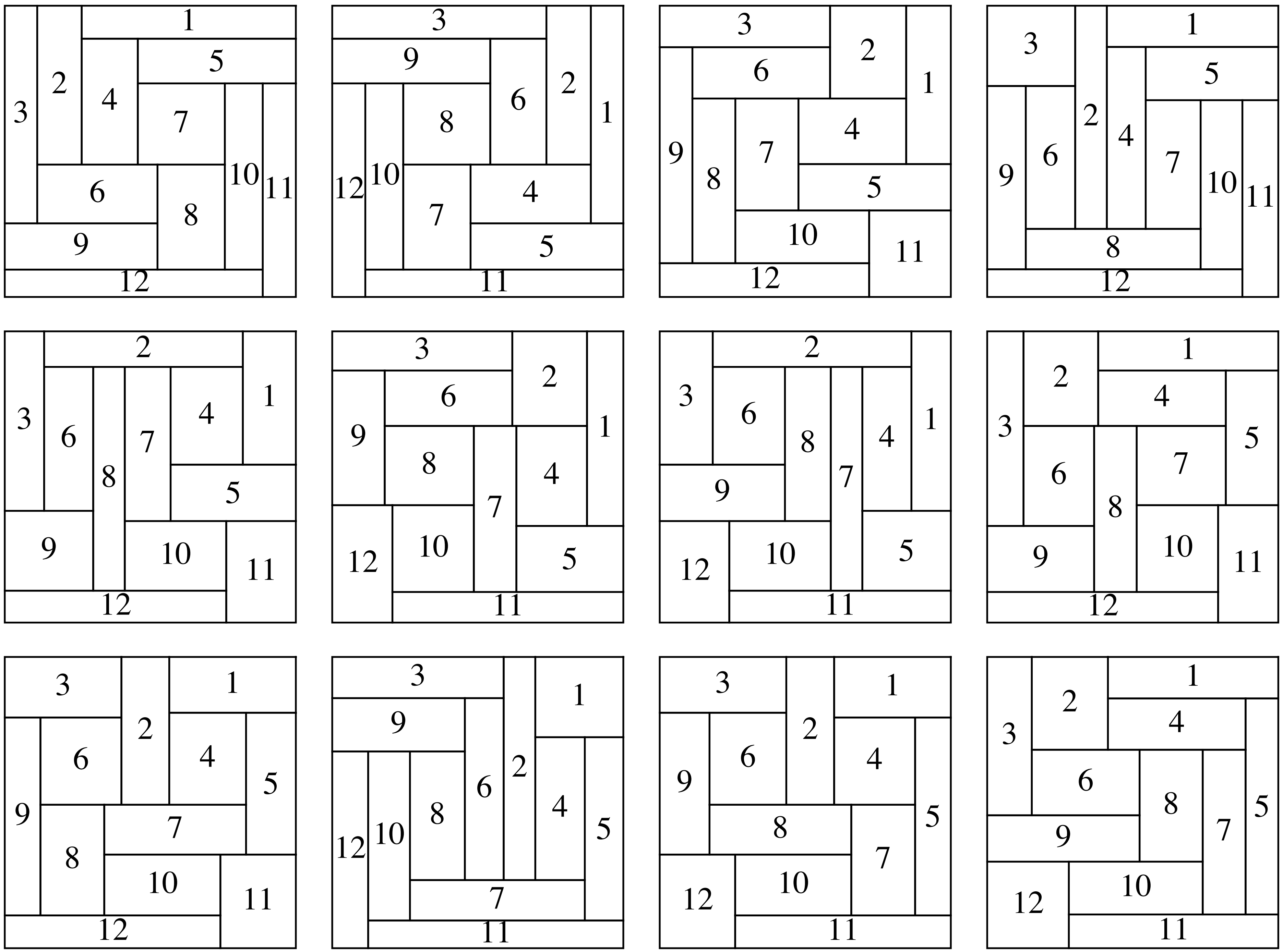
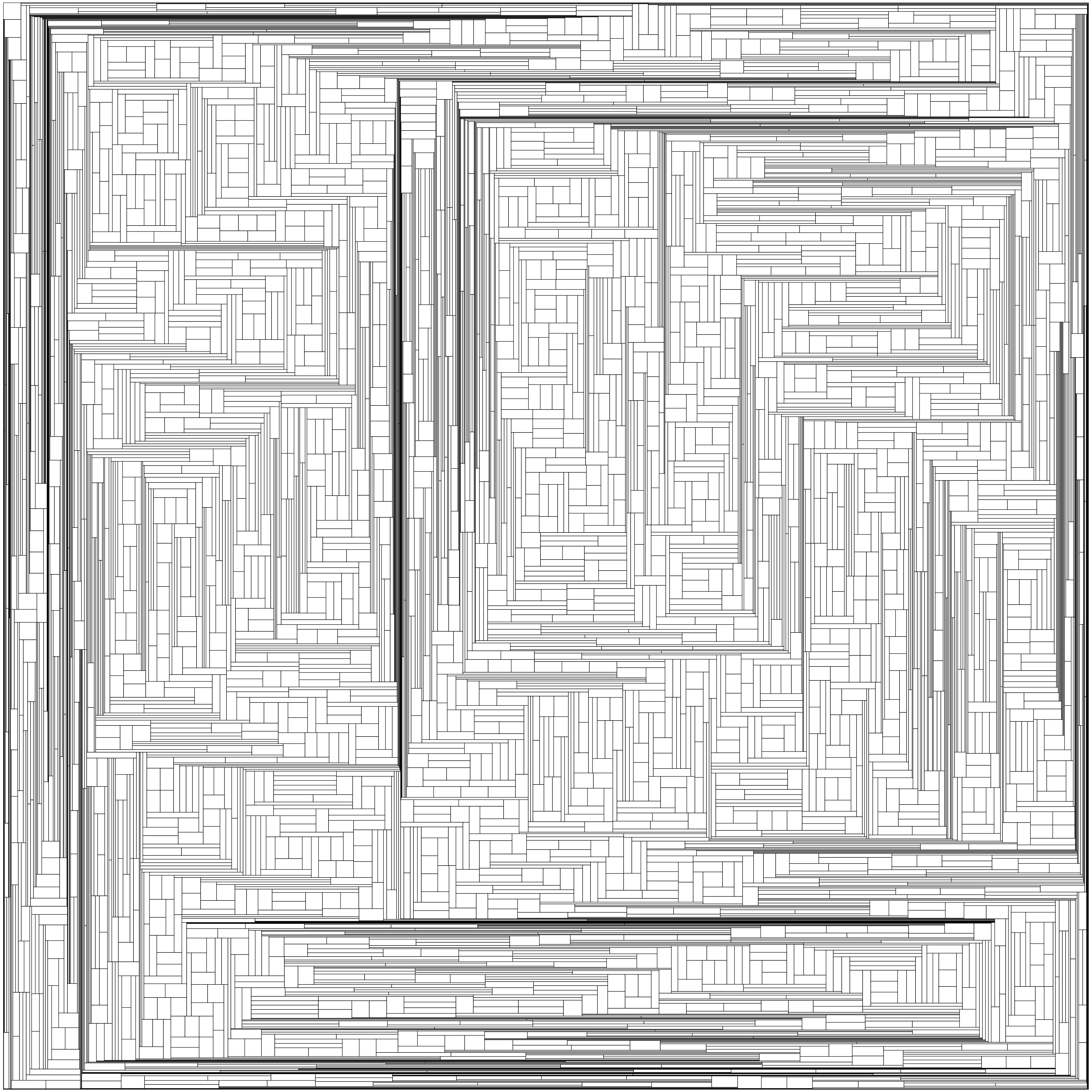
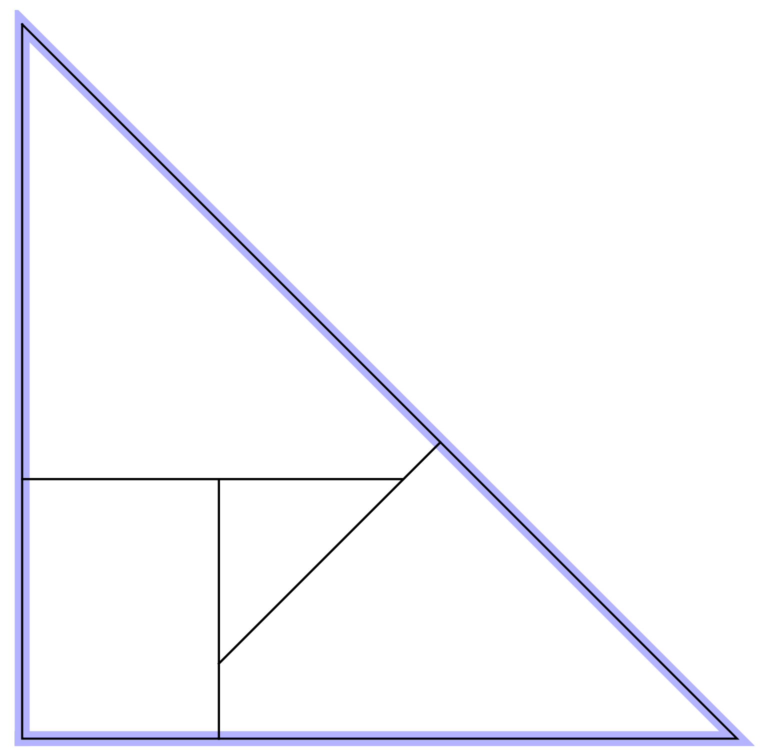


A random bipolar orientation of a random graph:

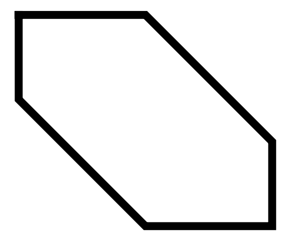


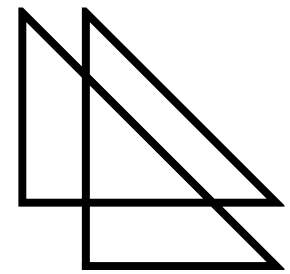
$$e^{\gamma h} dx^2 + e^{-\gamma h} dy^2 \quad ?$$

T-graphs with fixed slopes come in linear families (polytopes)



Polygons (or closed polygonal curves) with fixed edge slopes



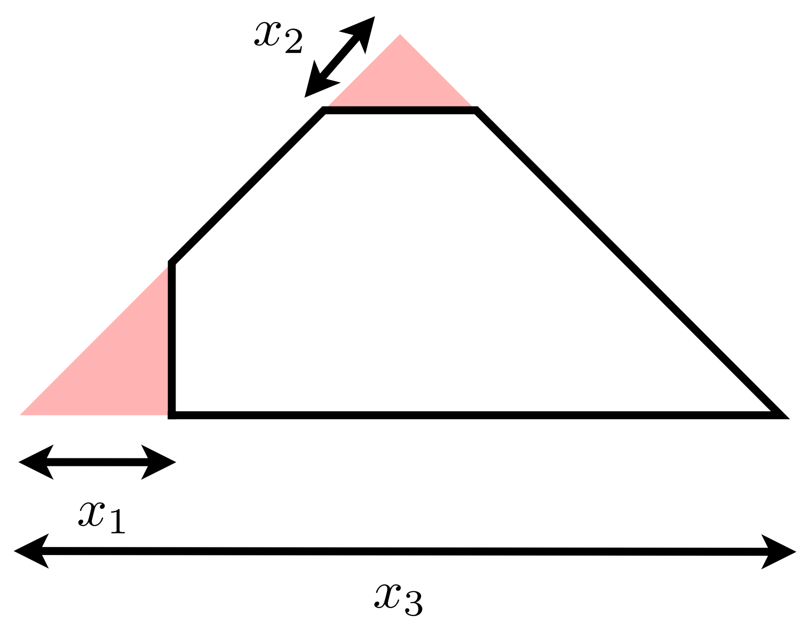


Thurston:

Given a convex n-gon, the space of closed polygonal curves with the same edge slopes is $\cong \mathbb{R}^{n-2}$.

On this space the signed area is a quadratic form of signature $(1, n-3)$.

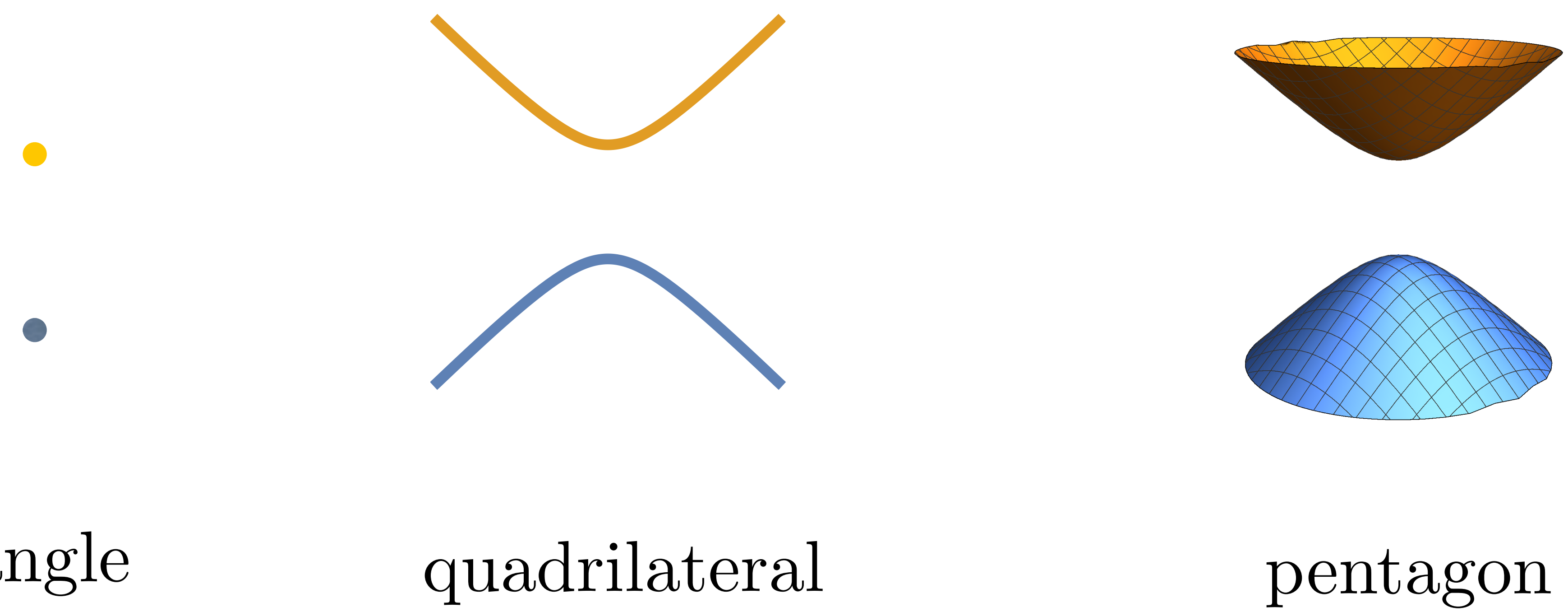
Proof by picture:



$\square$

$$A = C_3 x_3^2 - C_1 x_1^2 - C_2 x_2^2$$

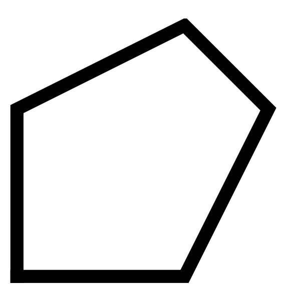
For fixed area, there are two components to the space, called orientations:

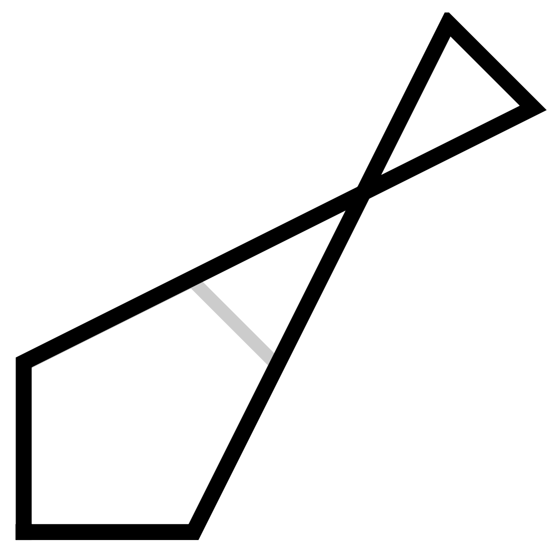


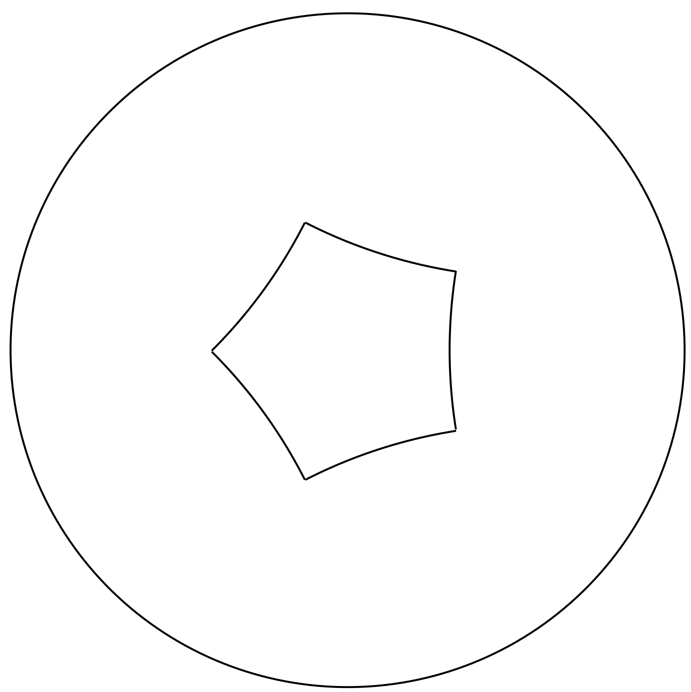
Fixing area$=1$, each component is isometric to $\mathbb{H}^{n-3}$.

The space of area-1 convex polygons is a convex polytope $R = R(P)$ in $\mathbb{H}^{n-3}$

"Butterfly moves" are hyperbolic isometries (reflections in the sides of R).





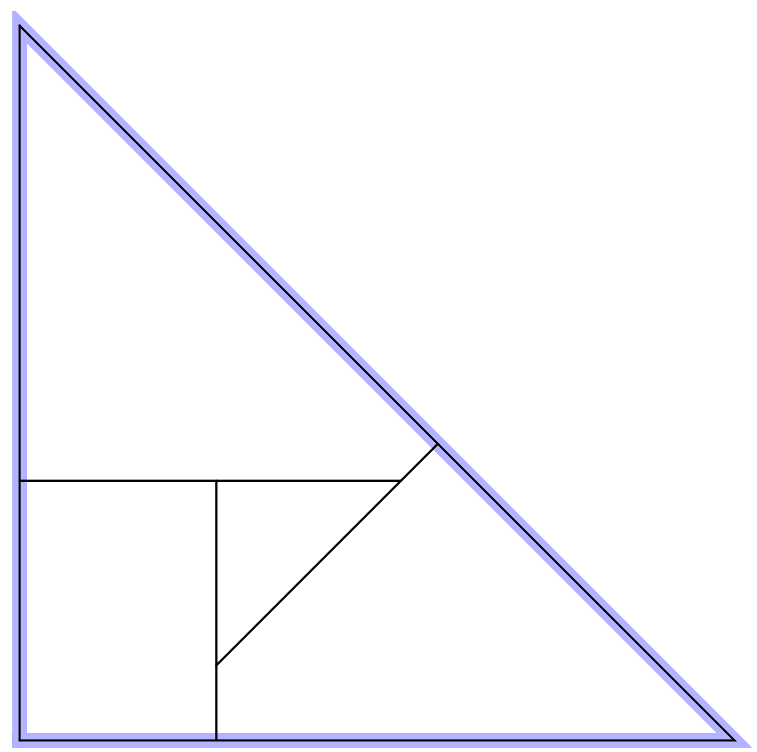


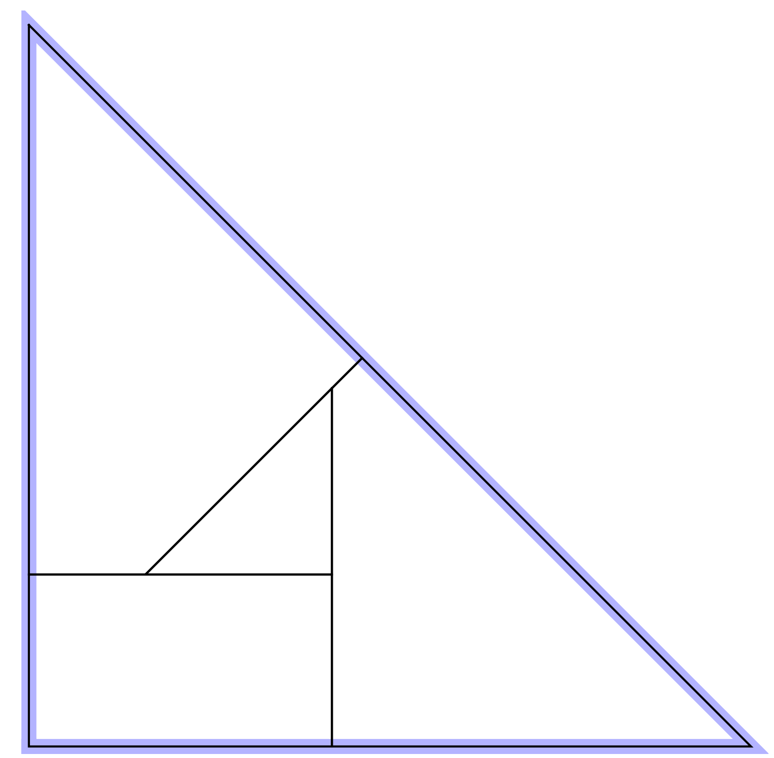
Shape of R depends on slopes of sides of P:

parallel sides of P implies side of R "at infinity".

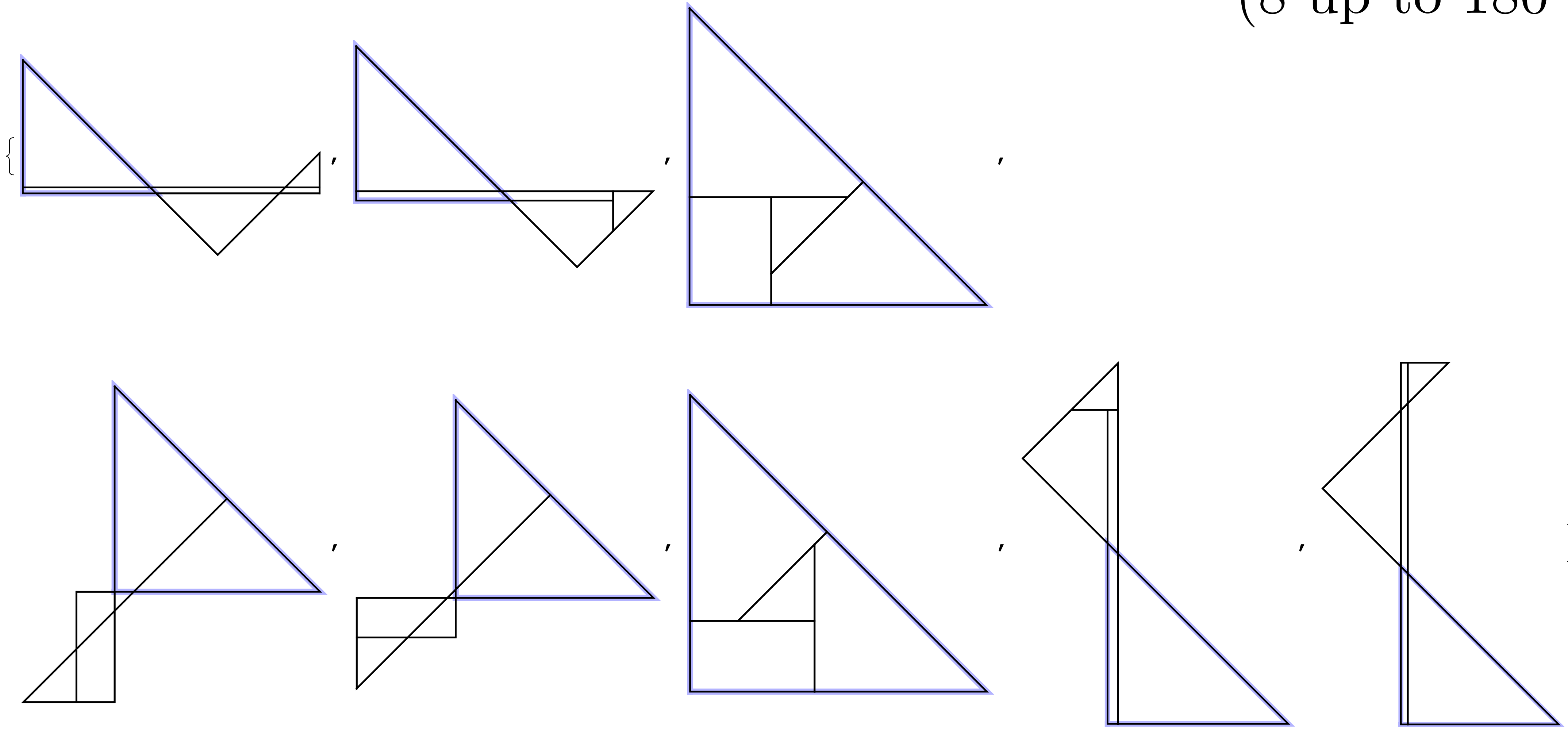
Fix a tiling family (t-graph with fixed combinatorics and slopes)

Thm: For generic slopes, there is exactly one (generalized) tiling for each choice of areas and tile orientations.





For example, if we fix the areas, in this case there are 16 generalized tilings (8 up to 180° rotation).



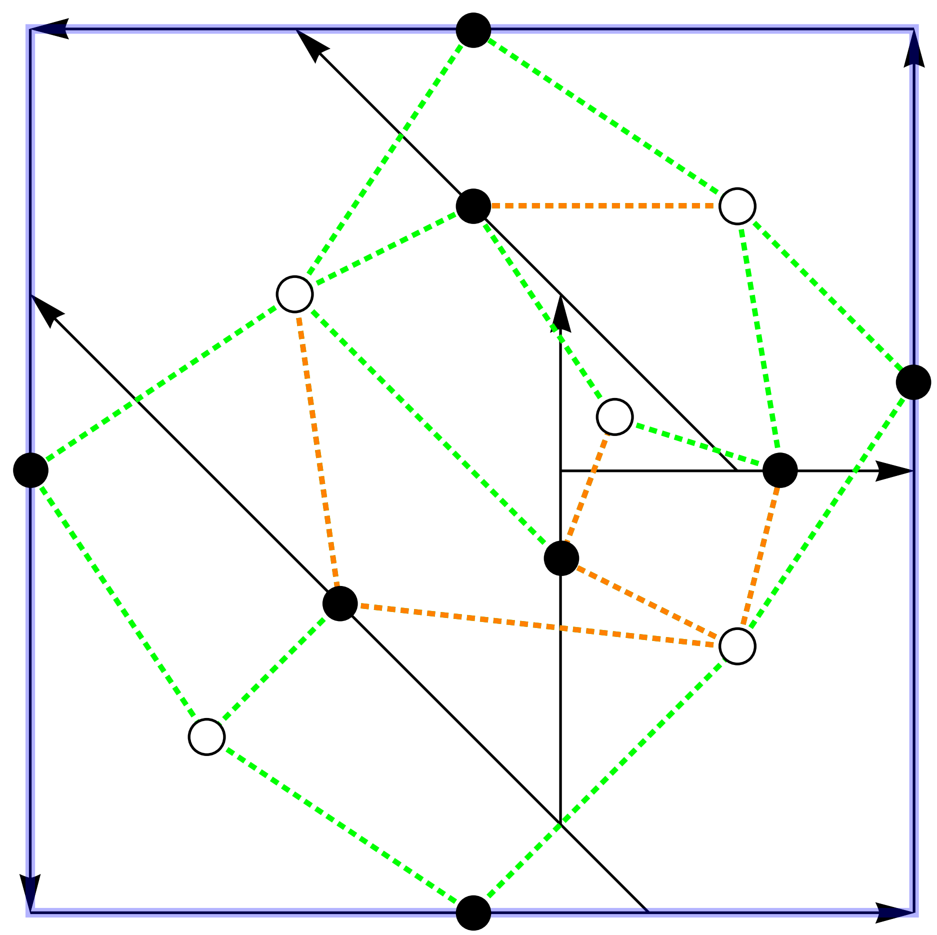
Reality conjecture:

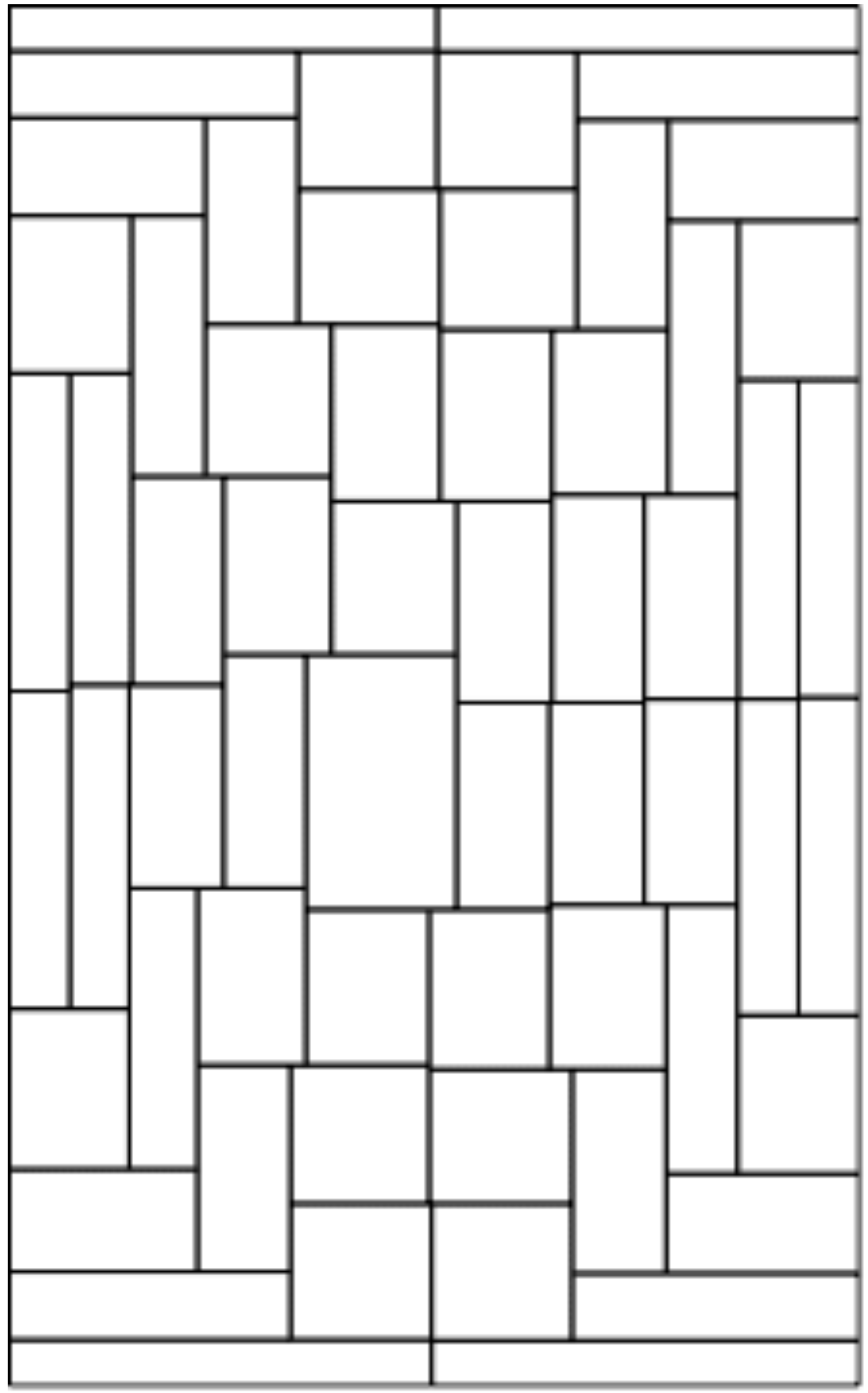
For rational slopes and areas, the Galois group permutes the solutions.

Thm: For each choice of orientation, the set of possible areas (if nonempty) is homeomorphic to a closed ball of dimension F.

Proof: The map $\Psi : \{\text{intercepts}\} \to \{\text{areas}\}$ is a local homeomorphism because $D\Psi$ is a Kasteleyn matrix for the underlying bipartite graph. (which has dimer covers!)

Injectivity of Ψ follows from convexity: given two tilings with same areas and same orientations, their average has greater area for each tile. $\square$





Conclusion:

for rectangulations, polytopes $\leftrightarrow$ bipolar orientations of network

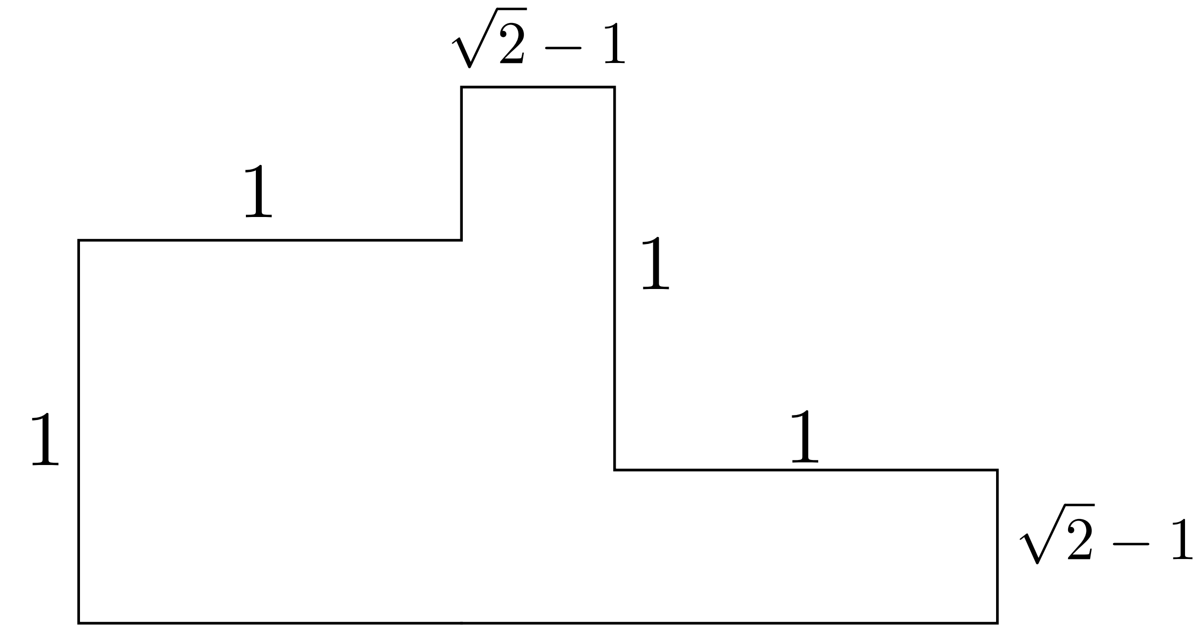
for generic slopes, polytopes $\leftrightarrow$ orientations (of white vertices)

Q. what about intermediate cases?

Many nontrivial facts can be proved using networks...

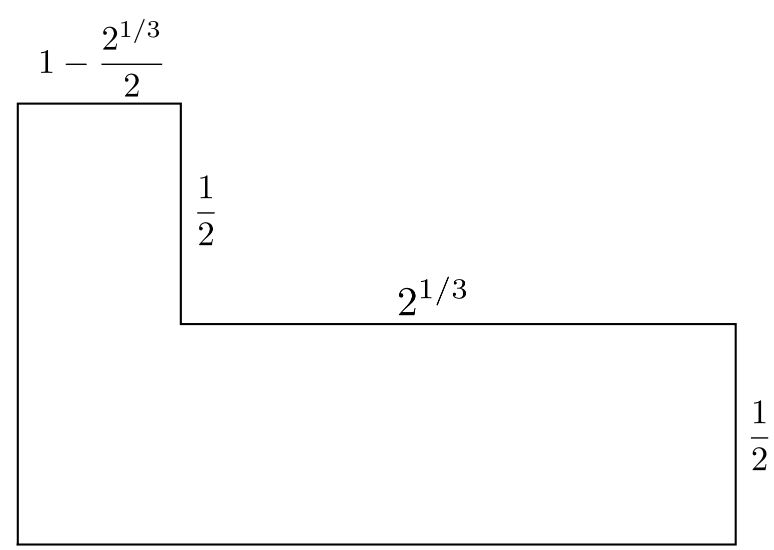
Q1. Can P be tiled with squares?

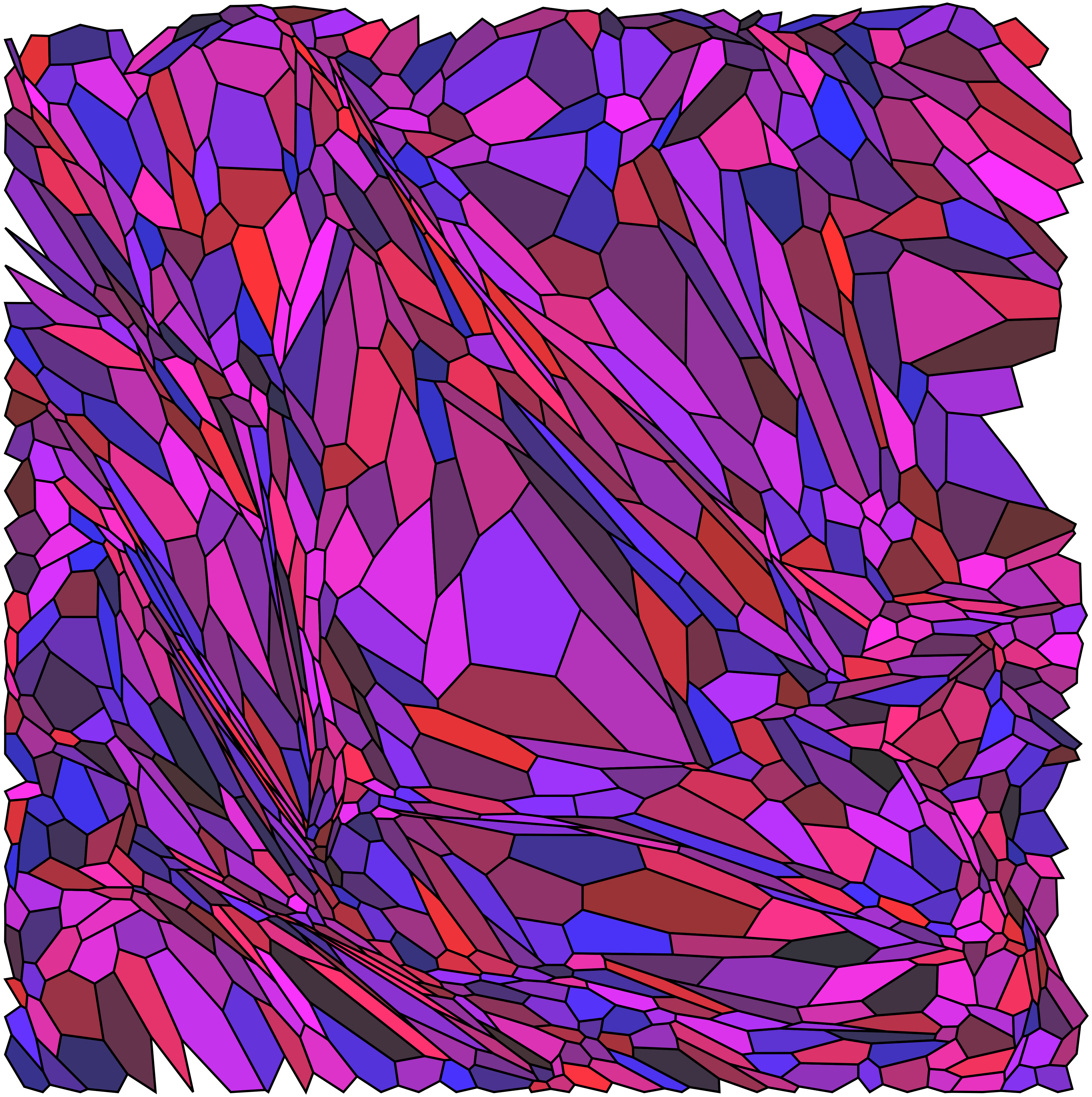
(no)



Q2. Can Q be tiled with rectangles of rational area?

(no)





thank you for your attention!